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Isotonic Approximation in L_s

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1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $0 < s < \infty$. Denote by $L_s(\Omega, \mathscr{A}, P)$ the system of all equivalence classes of $\mathscr{A}$-measurable functions X: $\Omega \to \overline{\mathbb{R}}$ with $\|X\|_s := [\int |X|^s \, dP]^{1/s} < \infty$. For $s > 1$ the space $L_s(\Omega, \mathscr{A}, P)$ endowed with the norm $\| \, \|_s$ is a uniformly convex Banach space.

Let $\mathscr{L} \subset \mathscr{A}$ be a σ-lattice, i.e., a system of sets which is closed under countable unions and intersections and contains $\varnothing$ and Ω. A function X: $\Omega \to \overline{\mathbb{R}}$ is $\mathscr{L}$-measurable if $\{X > a\} \in \mathscr{L}$ for all $a \in \mathbb{R}$. Denote by $L_s(\mathscr{L})$ the system of all equivalence classes in $L_s(\Omega, \mathscr{A}, P)$ containing an $\mathscr{L}$-measurable function. Then $L_s(\mathscr{L})$ is a closed convex set. Let $X \in L_s(\Omega, \mathscr{A}, P)$; an element $Y \in L_s(\mathscr{L})$ is called a *conditional s-mean* of X given the σ-lattice $\mathscr{L}$, if

$$\|X - Y\|_s = \min\{\|X - Z\|_s : Z \in L_s(\mathscr{L})\}.$$

For $s > 1$ the uniform convexity of L_s guarantees existence and uniqueness of a conditional s-mean of X given $\mathscr{L}$, denoted by $P_s^{\mathscr{L}} X$.

The above minimization problem is very important, since it allows one to treat approximation problems under order restrictions. For instance, let P be a probability measure on the Borel-field $\mathbb{B}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{M}$ be the family of all functions $Y \in L_s(\mathbb{R}^n, \mathbb{B}^n, P)$ which are monotone nondecreasing in each component. Define $\mathscr{L} := \{C \in \mathbb{B}^n : x \in C, x \leqslant y \Rightarrow y \in C\}$, where $x \leqslant y$ means $x_i \leqslant y_i$ for all components. Then $\mathscr{L}$ is a

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σ-lattice and $\mathscr{M}$ is the system of all $\mathscr{L}$-measurable $Y \in L_s(\mathbb{R}^n, \mathbb{B}^n, P)$. Hence for each $X \in L_s(\mathbb{R}^n, \mathbb{B}^n, P)$ the conditional s-mean of X given $\mathscr{L}$ is the unique function which is monotone nondecreasing in each component such that the $\| \|_s$-distance from X is minimal among all functions which are monotone non-decreasing in each component.

For $s = 2$, approximation problems of this type are highly relevant to the theory of isotonic regression. Illuminating examples and the broad theory of isotonic regression can be found in the book of Barlow *et al.* [3]. Although much work has been done for the case $s = 2$—the statistical inference under order restrictions—little is known for the case $s \neq 2$.

The concept of conditional s-means given a σ-lattice was introduced in Brunk [8]. For $s = 2$ it is the usual conditional expectation given a σ-lattice, defined on L_2 (see [5]). If $\mathscr{L}$ is a σ-field one obtains the s-predictors in the sense of Ando and Amemiya (see [1]), which coincide for $s = 2$ with the conditional expectations.

The theory developed here traces back to Kolmogoroff [13] and Wiener [20], it is intimately related to Wald's decision theory and plays an important role in Non-linear Prediction, Filtering, Regression and Bayes Estimation.

In a forthcoming paper we apply this theory to define and estimate a "natural" median (i.e., a "reasonable" selection of a median). At first we collect some properties of $P_s^{\mathscr{L}}$ on L_s and prove an integral inequality for the conditional s-means (Theorem 2.11.) which is essential for the further considerations (see Section 2).

In Section 3 we extend the operator $P_s^{\mathscr{L}}$ from L_s to L_{s-1} as the unique operator preserving the property of monotone continuity. An example shows that even for $\mathscr{L} = \{\emptyset, \Omega\}$ a monotone continuous extension to larger spaces L_r $(r < s - 1)$ is not possible in nearly all cases.

In Section 4 we show that $P_s^{\mathscr{L}_n} X$ converges P-a.e. to $P_s^{\mathscr{L}_\infty} X$ if $X \in L_{s-1}$ and the σ-lattices $\mathscr{L}_n$ increase or decrease to the σ-lattice $\mathscr{L}_\infty$. This result contains and extends other results in this direction. For $s = 2$ and σ-fields it is a classical martingale theorem and yields a result of Brunk and Johansen [9], for $s \neq 2$ and σ-fields it extends a result of Ando and Amemiya [1].

In Section 5 we prove maximal inequalities, i.e., inequalities for $P\{\sup_{n \in \mathbb{N}} |P_s^{\mathscr{L}_n} X| > a\}$ and $\int \sup_{n \in \mathbb{N}} |P_s^{\mathscr{L}_n} X|^r \, dP$. Our inequalities applied to $s = 2$ and σ-fields yield the classical maximal inequalities; applied to $s \neq 2$ and σ-fields they lead to much sharper maximal inequalities than those of Ando and Amemiya (see [1]). For σ-lattices they seem to be the first results in this direction.

In Section 6 we start with a characterization result for conditional s-means with respect to σ-fields (Theorem 6.1). This result, applied to $s = 2$, yields the characterization results for classical conditional expectations of Bahadur [2], Douglas [10], Moy [15] and Pfanzagl [17]. Then we characterize conditional s-means with respect to σ-lattices (Theorem 6.4). For $s \neq 2$ this is the first characterization result of conditional s-means given σ-lattices. For $s=2$ we prove a further characterization result (Theorem 6.3) which contains Dykstra's [11] characterization for conditional expectations given a σ-lattice.

2. Properties of $P_s^{\mathscr{L}} \mid L_s$

Let $(\Omega, \mathscr{A}, P)$ be a probability space. During this section let $\mathscr{L} \subset \mathscr{A}$ be a fixed σ-lattice and $1 < s < \infty$.

The conditional s-mean $T = P_s^{\mathscr{L}}$: $L_s \to L_s$ has, according to Brunk [8], the following properties

T is idempotent; i.e., $T(TX) = TX$; (2.0)

T is positive homogeneous; i.e., $T(aX) = aTX, \quad a \geqslant 0$; (2.1)

T is translation invariant; i.e., $T(X+b) = TX + b, \quad b \in \mathbb{R}$, (2.2)

and fulfills

$$\int (X - TX)^{\underline{s-1}} TX \, dP = 0; \tag{2.3}$$

$$\int (X - TX)^{\underline{s-1}} Z \, dP \leqslant 0 \qquad \text{if} \quad Z \in L_s(\mathscr{L}), \tag{2.4}$$

where $a^{\underline{s-1}} = |a|^{s-1} \operatorname{sign} a$ for $a \in \mathbb{R}$.

Relation (2.4) aplied to $Z \equiv +1$ and $Z \equiv -1$ implies that

$$T \text{ is } s\text{-expectation invariant; i.e., } \int (X - TX)^{\underline{s-1}} \, dP = 0. \tag{2.5}$$

Using that $X^{\underline{s-1}} TX \geqslant (X - TX)^{\underline{s-1}} TX$ with $>$ if $TX \neq 0$, we obtain from (2.3)

T is s-strictly monotonic at 0; i.e., (2.6)

$$\int X^{\underline{s-1}} TX \, dP > 0 \qquad \text{for} \quad TX \neq 0.$$

As $T = P_s^{\mathscr{L}}$ is a nearest point projection onto a closed convex set of the uniformly convex space L_s it is well known that

T is continuous in the norm topology of L_s. (2.7)

Brunk [8] has shown that $a \leqslant X \leqslant b$ implies $a \leqslant TX \leqslant b$. We show that

$$T \text{ is monotone.} \tag{2.8}$$

Let $X \leqslant Y$ and define $Z_1 = TX \wedge TY$, $Z_2 = TX \vee TY$. Applying Lemma 7.2(iii) pointwise to $a = Y$, $b = TY$, $c = X$, $d = TX$ we obtain by integration

$$\|Y - TY\|_s^s + \|X - TX\|_s^s \geqslant \|Y - Z_2\|_s^s + \|X - Z_1\|_s^s.$$

Since Z_1, $Z_2 \in L_s(\mathscr{L})$ we obtain $\|Y - TY\|_s = \|Y - Z_2\|_s$ and hence $Z_2 = TY$; i.e., $TX \leqslant TY$.

From (2.7) and (2.8) we obtain that

$$T \text{ is monotone continuous; i.e.,} \tag{2.9}$$

$$X_n \uparrow X \ (X_n \downarrow X) \text{ implies } TX_n \uparrow TX \ (TX_n \downarrow TX).$$

Let $\bar{\mathscr{L}} := \{\bar{C} : C \in \mathscr{L}\}$ be the dual σ-lattice of $\mathscr{L}$. Since $L_s(\mathscr{L}) = -L_s(\bar{\mathscr{L}})$ it follows directly from the definition of s-means that

$$P_s^{\bar{\mathscr{L}}} X = -P_s^{\mathscr{L}}(-X), \qquad X \in L_s. \tag{2.10}$$

Now we prove an integration inequality which is an important tool in the following sections. For $s = 2$ this was proven in [3, p. 342] by other techniques. Denote by $\mathbb{B}$ the Borel-field of $\mathbb{R}$.

THEOREM 2.11. *Let φ: $\mathbb{R} \to [0, \infty)$ be $\mathbb{B}$-measurable and Z: $\Omega \to \mathbb{R}$ be $\mathscr{L}$-measurable. Then we have for all $X \in L_s$*

$$\int (X - P_s^{\mathscr{L}} X)^{\underline{s-1}} Z \varphi \circ P_s^{\mathscr{L}} X \, dP \leqslant 0$$

if the integral exists.

Proof. W.l.o.g. we may assume that Z is bounded. By the usual techniques it can be seen that it suffices to prove the assertion for non-negative bounded functions φ with bounded derivative. Let such a φ be given and $M := \sup_{t \in \mathbb{R}} |\varphi'(t)| < \infty$.

(i) We prove the assertion for $Z = 1_C$ with $C \in \mathscr{L}$:

Let $Y_\alpha := P_s^{\mathscr{L}} X + (\alpha/M) \varphi \circ P_s^{\mathscr{L}} X$, $0 \leqslant \alpha < 1$. Then

$$Y_\alpha = \psi_\alpha \circ P_s^{\mathscr{L}} X, \tag{1}$$

where $\psi_\alpha(t) := t + (\alpha/M)\, \varphi(t)$, $t \in \mathbb{R}$, $0 \leqslant \alpha < 1$.

Since ψ_α: $\mathbb{R} \to \mathbb{R}$ is monotone increasing, we obtain from (1) that

$$Y_\alpha \text{ is } \mathscr{L}\text{-measurable for all } 0 \leqslant \alpha < 1. \tag{2}$$

Define

$$Z_\alpha := P_s^{\mathscr{L}} X + (\alpha/M)\, 1_C \varphi \circ P_s^{\mathscr{L}} X, \qquad 0 \leqslant \alpha < 1.$$

Since $\varphi \geqslant 0$ we obtain with (2) for all $b \in \mathbb{R}$, $0 \leqslant \alpha < 1$

$$\{Z_\alpha > b\} = \{P_s^{\mathscr{L}} X > b\} \cup (C \cap \{Y_\alpha > b\}) \in \mathscr{L}$$

and hence

$$Z_\alpha \text{ is } \mathscr{L}\text{-measurable for all } 0 \leqslant \alpha < 1. \tag{3}$$

The function

$$g(\alpha) := \|X - Z_\alpha\|_s^s, \qquad 0 \leqslant \alpha < 1$$

attains its minimum at $\alpha = 0$, because $Z_0 = P_s^{\mathscr{L}} X$ and $Z_\alpha \in L_s(\mathscr{L})$ for all $\alpha \in [0, 1)$. Hence

$$0 \leqslant g'(0) = -(s/M) \int (X - P_s^{\mathscr{L}} X)^{\underline{s-1}}\, 1_C \varphi \circ P_s^{\mathscr{L}} X \, dP,$$

which implies the assertion for $Z = 1_C$.

(ii) The assertion holds for all non-negative bounded $\mathscr{L}$-measurable functions by (i), since every non-negative $\mathscr{L}$-measurable function is the monotone limit of non-negative $\mathscr{L}$-measurable simple functions.

(iii) Now let Z be $\mathscr{L}$-measurable and bounded. Since $Z = Z^+ - Z^-$, it suffices, according to (ii), to prove that

$$-\int (X - P_s^{\mathscr{L}} X)^{\underline{s-1}} Z^- \varphi \circ P_s^{\mathscr{L}} X \, dP \leqslant 0. \tag{4}$$

The function $\psi(t) := \varphi(-t)$, $t \in \mathbb{R}$, is a non-negative bounded $\mathbb{B}$-measurable function. Hence (4) is fulfilled if we show

$$\int (-X - (-P_s^{\mathscr{L}} X))^{\underline{s-1}} Z^- \psi(-P_s^{\mathscr{L}} X) \, dP \leqslant 0. \tag{5}$$

Let $\bar{\mathscr{L}} = \{\bar{C} : C \in \mathscr{L}\}$ be the dual lattice of $\mathscr{L}$. Since Z^- is $\bar{\mathscr{L}}$-measurable and $-P_s^{\mathscr{L}} X = P_s^{\bar{\mathscr{L}}}(-X)$ by Property 2.10, (ii) applied to the σ-lattice $\bar{\mathscr{L}}$ yields (5).

3. Extension and Properties of $P_s^{\mathscr{L}}/L_{s-1}$

Up to now the operator $P_s^{\mathscr{L}}$ is defined on the domain L_s. For $s=2$ it is known that $P_2^{\mathscr{L}}$ can be extended from L_2 to L_1. It is our aim to extend for general $s>1$ the conditional s-mean $P_s^{\mathscr{L}}$ as a monotone continuous operator from L_s to larger spaces L_r, more exactly to choose r as small as possible. For instance, can we always extend the conditional s-mean from L_s as a monotone continuous operator mapping L_1 into L_1?

It turns out that we can extend $P_s^{\mathscr{L}}$ from L_s to L_{s-1} but, even in extremely simple cases, not beyond L_{s-1}. This shows, for instance, that for $s>2$ we cannot extend to L_1 but for $1<s<2$ we can extend even beyond L_1. We write $P_s^{\mathscr{L}}/L$ in order to indicate that $P_s^{\mathscr{L}}$ is defined on $L \subset L_{s-1}$.

Definition 3.1. (i) Let $\mathfrak{L}_{s-1}$ be the system of all $X \in L_{s-1}(\Omega, \mathscr{A}, P)$, which are bounded from below by a function of L_s. For $X \in \mathfrak{L}_{s-1}$ define

$$P_s^{\mathscr{L}}X := \lim_{n \in \mathbb{N}} P_s^{\mathscr{L}}(X \wedge n).$$

We remark that this limit exists since $P_s^{\mathscr{L}}/L_s$ is monotone and $X \wedge n$, $n \in \mathbb{N}$, is a nondecreasing sequence in L_s. It is easy to see (use Property (2.9)) that $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$ is a monotone extension of $P_s^{\mathscr{L}}/L_s$.

(ii) For $X \in L_{s-1}$ define

$$P_s^{\mathscr{L}}X := \lim_{n \in \mathbb{N}} P_s^{\mathscr{L}}(X \vee (-n)).$$

We remark that this limit exists since $P_s^{\mathscr{L}}|\mathfrak{L}_{s-1}$ is monotone and $X \vee (-n)$, $n \in \mathbb{N}$, is a non-increasing sequence in $\mathfrak{L}_{s-1}$. Since it turns out that $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$ is monotone continuous (this is proven in Theorem 3.2), $P_s^{\mathscr{L}}/L_{s-1}$ is an extension of $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$ and hence of $P_s^{\mathscr{L}}/L_s$.

Theorem 3.2. *Let $\mathscr{L} \subset \mathscr{A}$ be a σ-lattice and $1<s<\infty$. Then the operator $P_s^{\mathscr{L}}/L_{s-1}$ has the following properties:*

(P0) *$P_s^{\mathscr{L}}$ maps L_{s-1} into L_{s-1},*

(P1) *$P_s^{\mathscr{L}}/L_{s-1}$ is idempotent,*

(P2) *$P_s^{\mathscr{L}}/L_{s-1}$ is montone,*

(P3) *$P_s^{\mathscr{L}}/L_{s-1}$ is positive homogeneous,*

(P4) *$P_s^{\mathscr{L}}/L_{s-1}$ is translation invariant,*

(P5) *$P_s^{\mathscr{L}}/L_{s-1}$ is monotone continuous,*

(P6) *$P_s^{\mathscr{L}}/L_{s-1}$ is s-expectation invariant,* (P7) $\int (X - P_s^{\mathscr{L}} X)^{\underline{s-1}} Z\varphi \circ P_s^{\mathscr{L}} X \, dP \leqslant 0$, *if* $X \in L_{s-1}$, Z: $\Omega \to \mathbb{R}$ *is* $\mathscr{L}$-*measurable*, φ: $\mathbb{R} \to [0, \infty)$ *is* $\mathbb{B}$-*measurable and the integral exists,*

(P8) $\int (X - P_s^{\mathscr{L}} X)^{\underline{s-1}} \varphi \circ P_s^{\mathscr{L}} X \, dP = 0$ *if* $X \in L_{s-1}$, φ: $\mathbb{R} \to \mathbb{R}$ *is* $\mathbb{B}$-*measurable and the integral exists.*

Proof. We prove the properties above for $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$, using the corresponding properties of $P_s^{\mathscr{L}}/L_s$. Using the then proven properties of $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$, one obtains in a similar way the stated properties of $P_s^{\mathscr{L}}/L_{s-1}$.

(P0) Let $X \in \mathfrak{L}_{s-1}$. Then $X_n := X \wedge n \in L_s$. As $P_s^{\mathscr{L}}/L_s$ is s-expectation invariant (see Property 2.5) we obtain

$$P[(X_n - P_s^{\mathscr{L}} X_n)^{\underline{s-1}}] = 0 \tag{1}$$

As $(X_n - P_s^{\mathscr{L}} X_n)^{\underline{s-1}} \leqslant (X - P_s^{\mathscr{L}} X_n)^{\underline{s-1}}$ and $X \in \mathfrak{L}_{s-1}$, $P_s^{\mathscr{L}} X_n \in L_s$, (1) implies

$$0 \leqslant P[(X - P_s^{\mathscr{L}} X_n)^{\underline{s-1}}] < \infty, \qquad n \in \mathbb{N} \tag{2}$$

Since $(X - P_s^{\mathscr{L}} X_n)^{\underline{s-1}} \downarrow (X - P_s^{\mathscr{L}} X)^{\underline{s-1}}$, (2) implies

$$0 \leqslant P[(X - P_s^{\mathscr{L}} X)^{\underline{s-1}}] < \infty$$

and hence $X - P_s^{\mathscr{L}} X \in L_{s-1}$. As L_{s-1} is a linear space and $X \in L_{s-1}$, consequently $P_s^{\mathscr{L}} X \in L_{s-1}$. Since $X \geqslant Y \in L_s$ implies $P_s^{\mathscr{L}} X \geqslant P_s^{\mathscr{L}} Y$, we get $P_s^{\mathscr{L}} X \in \mathfrak{L}_{s-1}$.

Property (P1) follows directly from (2.0) and the definition of $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$.

Property (P2) follows from (2.8) and the definition of $P_s^{\mathscr{L}}/\mathfrak{L}_{s-1}$.

Property (P7) W.l.o.g. we may assume that Z is bounded. By the usual techniques it can be seen that if suffices to prove the assertion for non-negative, continuous and bounded functions φ. Let now $X \in \mathfrak{L}_{s-1}$ and $X_n := X \wedge n$. Then $X_n \in L_s$ and therefore Theorem 2.11 implies

$$\int (X_n - P_s^{\mathscr{L}} X_n)^{\underline{s-1}} Z\varphi \circ P_s^{\mathscr{L}} X_n \, dP \leqslant 0, \qquad n \in \mathbb{N}.$$

As φ is continuous, $\varphi \circ P_s^{\mathscr{L}} X_n \to \varphi \circ P_s^{\mathscr{L}} X$ by definition of $P_s^{\mathscr{L}} X$. Since $P_s^{\mathscr{L}} X \in L_{s-1}$ according to (P0), it is easy to see that

$$|(X_n - P_s^{\mathscr{L}} X_n)^{\underline{s-1}} Z\varphi \circ P_s^{\mathscr{L}} X_n|, \qquad n \in \mathbb{N},$$

is bounded by an integrable function. Hence the Theorem of Lebesgue implies the assertion.

Property (P6) follows directly from (P7).

Property (P5) W.l.o.g. we prove only the decreasing case. Let $X_n \in \mathrm{L}_{s-1}$ with $X_n \downarrow X \in \mathfrak{L}_{s-1}$. Then by (P6)

$$\int (X_n - P_s^{\mathscr{S}} X_n)^{\underline{s-1}}\, dP = 0, \qquad n \in \mathbb{N}. \tag{3}$$

By (P2) we obtain $P_s^{\mathscr{S}} X \leqslant P_s^{\mathscr{S}} X_n \leqslant P_s^{\mathscr{S}} X_1$. Hence the Theorem of Lebesgue implies by (3)

$$\int \left(X - \lim_{n \in \mathbb{N}} P_s^{\mathscr{S}} X_n\right)^{\underline{s-1}} dP = 0. \tag{4}$$

As $\int (X - P_s^{\mathscr{S}} X)^{\underline{s-1}}\, dP = 0$ and $P_s^{\mathscr{S}} X \leqslant \lim_{n \in \mathbb{N}} P_s^{\mathscr{S}} X_n$, (4) implies that $P_s^{\mathscr{S}} X = \lim_{n \in \mathbb{N}} P_s^{\mathscr{S}} X_n$.

Properties (P3) and (P4) follow immediately from (P5), using the corresponding properties of $P_s^{\mathscr{S}} | L_s$ (see Section 2).

Property (P8) follows from (P7) applied to φ^+ and φ^- with $Z \equiv 1$ and $Z \equiv -1$.

The preceding theorem shows that there exists a monotone continuous extension of $P_s^{\mathscr{S}}/L_s$ which maps L_{s-1} into L_{s-1}. The following remark shows that, even for $\mathscr{S} = \{\varnothing, \Omega\}$, such an extension is not possible for any larger space L_r $(r < s-1)$.

Remark 3.3. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $1 < s < \infty$. Assume that $L_r \neq L_{s-1}$ for some $0 < r < s-1$. Let $X \geqslant 0$ with $X \in L_r - L_{s-1}$ be given and choose $\mathscr{S} = \{\varnothing, \Omega\}$. We shall show that each monotone continuous extension of $P_s^{\mathscr{S}}/L_s$ to L_r necessarily maps X to the function $\equiv \infty \notin L_r$.

Let $X_n := X \wedge n \in L_s$. Then $X_n \uparrow X$. If there exists a monotone continuous extension from $P_s^{\mathscr{S}}/L_s$ to L_r, then $c_n = P_s^{\mathscr{S}} X_n \uparrow P_s^{\mathscr{S}} X = c$. Assume that $c < \infty$. Since $P_s^{\mathscr{S}}/L_s$ is s-expectation invariant we have

$$0 = \int (X_n - c_n)^{\underline{s-1}}\, dP \geqslant \int (X_n - c)^{\underline{s-1}}\, dP > -\infty.$$

Since $X_n \uparrow X$, this implies $0 \geqslant \int (X - c)^{\underline{s-1}}\, dP > -\infty$. Hence $X \in L_{s-1}$, contradicting our assumption.

For each $X \in L_{s-1}$ we give now a characterization of the conditional s-mean $P_s^{\mathscr{S}} X$ as the unique $\mathscr{S}$-measurable function fulfilling two integral-inequalities. For the case $s = 2$ this is known for $X \in L_2$ (see [3]); it seems to be unknown even for $s = 2$ if $X \in L_1$.

THEOREM 3.4. *Let $X \in L_{s-1}$ and φ be a strictly increasing function such that $\varphi \circ P_s^{\mathscr{L}} X$ is bounded. Then $Y = P_s^{\mathscr{L}} X$ if and only if $Y \in L_{s-1}(\mathscr{L})$, $\varphi \circ Y$ is bounded and*

(i) $\int (X - Y)^{\underline{s-1}} Z \, dP \leqslant 0$ *for all bounded $\mathscr{L}$-measurable Z;*

(ii) $\int (X - Y)^{\underline{s-1}} \varphi \circ Y \, dP = 0$.

Proof. The direction "only if" follows from (P7) and (P8) of Theorem 3.2. For the converse direction it suffices to prove that if Y_1, $Y_2 \in L_{s-1}(\mathscr{L})$ are two functions fulfilling (i) and (ii), then $Y_1 = Y_2$.

By (ii) we have

$$\int (X - Y_j)^{\underline{s-1}} \varphi \circ Y_j \, dP = 0, \qquad j = 1, 2.$$

Since $\varphi \circ Y_j$, $j = 1, 2$ are bounded $\mathscr{L}$-measurable functions we obtain, together with (i),

$$\int ((Y_1 - X)^{\underline{s-1}} + (X - Y_2)^{\underline{s-1}})(\varphi \circ Y_1 - \varphi \circ Y_2) \, dP \leqslant 0.$$

Since φ is increasing, it is easy to see that

$$((Y_1 - X)^{\underline{s-1}} + (X - Y_2)^{\underline{s-1}})(\varphi \circ Y_1 - \varphi \circ Y_2) \geqslant 0.$$

Hence $\varphi \circ Y_1 = \varphi \circ Y_2$ P-a.e., whence $Y_1 = Y_2$ P-a.e.

COROLLARY 3.5. *Let $X \in L_{s-1}$. Then $Y = P_s^{\mathscr{L}} X$ if and only if $Y \in L_{s-1}(\mathscr{L})$ and*

(i) $\int_C (X - Y)^{\underline{s-1}} \, dP \leqslant 0$ *for all $C \in \mathscr{L}$;*

(ii) $\int_{\{Y \geqslant a\}} (X - Y)^{\underline{s-1}} \, dP = 0$ *for all $a \in \mathbb{R}$.*

Proof. The direction "only if" follows from (P7) and (P8) of Theorem 3.2. For the converse direction we have to prove (i) and (ii) of Theorem 3.4.

Since $\nu(B) := \int (X - Y)^{\underline{s-1}} 1_B \circ Y \, dP$, $B \in \mathbb{B}$, is a signed measure which is zero on the system $\{[a, \infty), a \in \mathbb{R}\}$ according to (ii), it is zero on $\mathbb{B}$ also. This implies (ii) of Theorem 3.4 even for all measurable functions φ, for which the integral in (ii) exists. Since every non-negative bounded $\mathscr{L}$-measurable function is the monotone limit of non-negative $\mathscr{L}$-measurable simple functions, (i) implies, that condition (i) of Theorem 3.4 holds for all non-negative, bounded, $\mathscr{L}$-measurable functions Z. Let Z be a bounded $\mathscr{L}$-

measurable function. Then $Z + a$ is a non-negative bounded $\mathscr{L}$-measurable function for suitable a and hence

$$\int (X - Y)^{\underline{s-1}} (Z + a)\, dP \leqslant 0.$$

Consequently we obtain (i) of Theorem 3.4.

Now we give some further properties of the extended $P_s^{\mathscr{L}}$. Since $L_s(\mathscr{L}) = -L_s(\bar{\mathscr{L}})$, we obtain from Property (P7) of Theorem 3.2 with $Z = -1_D$ that for $X \in L_{s-1}$

(P9) $\int_{C \cap D} (X - P_s^{\mathscr{L}} X)^{\underline{s-1}}\, dP \geqslant 0$ if $D \in \bar{\mathscr{L}}$, $C \in (P_s^{\mathscr{L}} X)^{-1} \mathbb{B}$, where $\mathbb{B}$ denotes the Borel-field of $\mathbb{R}$.

Using the monotone continuity of $P_s^{\mathscr{L}} / L_{s-1}$, Property 2.10 implies

(P10) $P_s^{\bar{\mathscr{L}}} X = -P_s^{\mathscr{L}}(-X), \quad X \in L_{s-1}$.

Since $X \in L_{s-1}$ and $Y = P_s^{\mathscr{L}} X$ fulfill (i) and (ii) of Corollary 3.5, we obtain for each $a \leqslant 1$ that $X_0 = X - a P_s^{\mathscr{L}} X$ and the $\mathscr{L}$-measurable function $Y_0 = (1 - a) P_s^{\mathscr{L}} X$ fulfill (i) and (ii), too. Hence Corollary 3.5 implies:

(P11) $P_s^{\mathscr{L}}(X - a P_s^{\mathscr{L}} X) = (1 - a) P_s^{\mathscr{L}} X, \quad a \leqslant 1, X \in L_{s-1}$.

Since $P_s^{\mathscr{L}}$ is monotone we have $P_s^{\mathscr{L}}(-|X|) \leqslant P_s^{\mathscr{L}} X \leqslant P_s^{\mathscr{L}} |X|$. Hence (P10) implies

(P12) $|P_s^{\mathscr{L}} X| \leqslant \max(P_s^{\mathscr{L}} |X|, P_s^{\bar{\mathscr{L}}} |X|), \quad X \in L_{s-1}$.

We remark that for σ-lattices it is not true that $|P_s^{\mathscr{L}} X| \leqslant P_s^{\mathscr{L}} |X|$; this can be seen by simple examples even in the case $s = 2$. Using (P2) and (P8) it is easy to see that for each interval $I \subset \mathbb{R}$

(P13) $X \in I$ P-a.e. implies $P_s^{\mathscr{L}} X \in I$ P-a.e.

Now we prove a convexity inequality for conditional s-means.

(P14) Let $I \subset \mathbb{R}$ be an interval and $\varphi\colon I \to \mathbb{R}$ be a non-decreasing continuous and convex function. If X, $\varphi \circ X \in L_{s-1}$ and $X \in I$ P-a.e. then

$$\varphi \circ P_s^{\mathscr{L}} X \leqslant P_s^{\mathscr{L}}(\varphi \circ X).$$

Proof. Since φ is continuous and convex there exist a_n, $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\varphi(x) = \sup_{n \in \mathbb{N}} (a_n x + b_n) \qquad \text{for all} \quad x \in I.$$

Since φ is non-decreasing, w.l.o.g. $a_n \geqslant 0$. From (P2), (P3), (P4) and (P13) we obtain

$$P_s^{\mathscr{L}}(\varphi \circ X) \geqslant \sup_{n \in \mathbb{N}} a_n P_s^{\mathscr{L}} X + b_n = \varphi \circ P_s^{\mathscr{L}} X. \qquad \text{Q.E.D.}$$

Finally we prove for each $r \geqslant s-1$

(P15) $X \in L_r \Rightarrow P_s^{\mathscr{L}} X \in L_r$.

According to (P12), we may w.l.o.g. assume that $X \geqslant 0$. Since $X^{r/(s-1)} \in L_{s-1}$, (P0) implies $P_s^{\mathscr{L}}(X^{r/(s-1)}) \in L_{s-1}$. As, by (P14) $(P_s^{\mathscr{L}} X)^{r/(s-1)} \leqslant P_s^{\mathscr{L}}(X^{r/(s-1)}) \in L_{s-1}$, we obtain $P_s^{\mathscr{L}} X \in L_r$.

4. Convergence a.e. of $P_s^{\mathscr{L}_n} X$

In this section we give a general a.e. convergence theorem for conditional s-means $P_s^{\mathscr{L}_n} X$ for all $X \in L_{s-1}$. For the case $s=2$, it is a result of [9]. For the special case of σ-fields and for $X \in L_s$ this was proven in [1]. The techniques used here, are related to the methods of Sparre *et al.* [18] and Brunk and Johansen [9].

If $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N} \cup \{\infty\}$, are σ-lattices, $\mathscr{L}_n$, $n \in \mathbb{N}$, increases [decreases] to $\mathscr{L}_\infty$ if $\mathscr{L}_n \subset \mathscr{L}_{n+1} [\mathscr{L}_n \supset \mathscr{L}_{n+1}]$ and $\mathscr{L}_\infty$ is the σ-lattice generated by $\bigcup_{n \in \mathbb{N}} \mathscr{L}_n [\mathscr{L}_\infty = \bigcap_{n \in \mathbb{N}} \mathscr{L}_n]$.

THEOREM 4.1. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $1 < s < \infty$. Let $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, be σ-lattices increasing or decreasing to the σ-lattice $\mathscr{L}_\infty$. Then*

$$P_s^{\mathscr{L}_n} X \to P_s^{\mathscr{L}_\infty} X, \qquad P\text{-a.e.},$$

for all $X \in L_{s-1}(\Omega, \mathscr{A}, P)$.

Proof. We prove only the decreasing case, which is somewhat more complicated. The proof for the increasing case runs similarly.

Let $X_n = P_s^{\mathscr{L}_n} X$, $n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Let $\bar{\mathscr{L}}_n := \{\bar{C} : C \in \mathscr{L}_n\}$. According to properties (P7) and (P9) of Section 3, we have

$$\int_{\{X_n < \alpha\} \cap C_n} (X-\alpha)^{\underline{s-1}} \, dP \leqslant 0 \qquad \text{if} \quad \alpha \in \mathbb{R}, C_n \in \mathscr{L}_n, n \in \bar{\mathbb{N}}, \tag{1}$$

and

$$\int_{\{X_n > \beta\} \cap D_n} (X-\beta)^{\underline{s-1}} \, dP \geqslant 0 \qquad \text{if} \quad \beta \in \mathbb{R}, D_n \in \bar{\mathscr{L}}_n, n \in \bar{\mathbb{N}}. \tag{2}$$

Define $\underline{X} = \underline{\lim}_{n \in \mathbb{N}} X_n$ and $\bar{X} = \overline{\lim}_{n \in \mathbb{N}} X_n$; then $\underline{X}, \bar{X}$ are $\mathscr{L}_\infty$-measurable. Let $C \in \mathscr{L}_\infty$ and $a \in \mathbb{R}$ be fixed and choose a sequence $\varepsilon_m \downarrow 0$. Define for $m \leqslant n \leqslant r$

$$C_{n,r}^m := C \cap \{X_n < a + \varepsilon_m, X_{n+1} \geqslant a + \varepsilon_m, \ldots, X_r \geqslant a + \varepsilon_m\}.$$

According to (1), we have

$$\int_{C^m_{n,r}} (X-(a+\varepsilon_m))^{\underline{s-1}}\, dP \leqslant 0. \tag{3}$$

Since $\sum_{n=m}^{r} C^m_{n,r} = C \cap \{\inf_{m \leqslant i \leqslant r} X_i < a+\varepsilon_m\} =: C^m_r$ and $C^m_r \uparrow_{r\to\infty} C^m := C \cap \{\inf_{i \geqslant m} X_i < a+\varepsilon_m\}$ we obtain, from (3),

$$\int_{C^m} (X-(a+\varepsilon_m))^{\underline{s-1}}\, dP \leqslant 0, \qquad m \in \mathbb{N}. \tag{4}$$

Since $1_{C^m} \to_{m\in\mathbb{N}} 1_{C\cap\{\underline{X} \leqslant a\}}$, (4) implies, using the Theorem of Lebesgue

$$\int_{\{\underline{X} \leqslant a\}\cap C} (X-a)^{\underline{s-1}}\, dP \leqslant 0, \qquad \text{if} \quad a \in \mathbb{R}, C \in \mathscr{L}_\infty. \tag{5}$$

In the same way we obtain

$$\int_{\{\bar{X} \geqslant b\}\cap D} (X-b)^{\underline{s-1}}\, dP \geqslant 0, \qquad \text{if} \quad b \in \mathbb{R}, D \in \bar{\mathscr{L}}_\infty. \tag{6}$$

Now we show, using the relations (1), (2), (5), (6) that

$$X_\infty \leqslant \underline{X}, \qquad \bar{X} \leqslant X_\infty, \qquad P\text{-a.e.}$$

This directly implies the assertion. We prove only $X_\infty \leqslant \underline{X}$ P-a.e. The proof for $\bar{X} \leqslant X_\infty$ runs similarly.

To prove $X_\infty \leqslant \underline{X}$ P-a.e., it suffices to show that for all a, $b \in \mathbb{R}$ with $a < b$ the set

$$E := \{\underline{X} \leqslant a < b < X_\infty\}$$

has P-measure zero. Relation (2) applied to $n = \infty$ and $D_\infty = \{\underline{X} \leqslant a\} \in \bar{\mathscr{L}}_\infty$ yields

$$\int_E (X-b)^{\underline{s-1}}\, dP \geqslant 0. \tag{7}$$

Relation (5), applied to $C = \{X_\infty > b\} \in \mathscr{L}_\infty$ yields

$$\int_E (X-a)^{\underline{s-1}}\, dP \leqslant 0. \tag{8}$$

Since $(X-a)^{\underline{s-1}} > (X-b)^{\underline{s-1}}$ on E, (7) and (8) imply $P(E) = 0$.

5. Maximal Inequalities and Norm Convergence of $P_s^{\mathscr{L}_n}X$

In this section we prove two maximal inequalities and give an application to norm convergence.

THEOREM 5.1. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $1 < s < \infty$. Let $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, be an increasing or decreasing sequence of σ-lattices. For non-negative $X \in L_{s-1}$ we have*

$$P\left\{\sup_{n\in\mathbb{N}} P_s^{\mathscr{L}_n}X > a\right\} \leqslant \frac{2^{|s-2|}}{a^{s-1}} \int_{\{\sup_{n\in\mathbb{N}} P_s^{\mathscr{L}_n}X > a\}} X^{s-1}\, dP, \qquad a > 0.$$

Proof. Let $X_n = P_s^{\mathscr{L}_n}X$, $n \in \mathbb{N}$. It suffices to prove for each $n \in \mathbb{N}$ that

$$P\left\{\max_{1\leqslant i\leqslant n} X_i > a\right\} \leqslant \frac{2^{|s-2|}}{a^{s-1}} \int_{\{\max_{1\leqslant i\leqslant n} X_i > a\}} X^{s-1}\, dP, \qquad a > 0. \tag{1}$$

As for decreasing sequences $\mathscr{L}_n$, $n \in \mathbb{N}$, the finite sequence $\mathscr{L}_n, \mathscr{L}_{n-1},..., \mathscr{L}_1$ is increasing; it suffices to prove (1) for the increasing case.

At first we show that

$$\int_{\{\max_{1\leqslant i\leqslant n} X_i > a\}} (X-a)^{\underline{s-1}}\, dP \geqslant 0, \qquad a > 0. \tag{2}$$

Let $A_i = \{X_1 \leqslant a,..., X_{i-1} \leqslant a,\ X_i > a\}$, $i = 1,..., n$. Since $A_i = \{X_i > a\} \cap D_i$, where $D_i \in \overline{\mathscr{L}_i}$ we obtain from Property (P9) of Section 3

$$\int_{A_i} (X-a)^{\underline{s-1}}\, dP \geqslant \int_{A_i} (X-X_i)^{\underline{s-1}}\, dP \geqslant 0.$$

As $\sum_{i=1}^n A_i = \{\max_{1\leqslant i\leqslant n} X_i > a\}$, this implies (2). Applying Lemma 7.2(i) and (ii) pointwise to $x = X(\omega)$ for all $\omega \in \{\max_{1\leqslant i\leqslant n} X_i > a\}$, we obtain by integration, using (2), inequality (1)

COROLLARY 5.2. *Let $(\Omega, \mathscr{A}, P)$ be a probability space, $1 < s < \infty$ and $r > s-1$. Let $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, be an increasing or decreasing sequence of σ-lattices. Then we have for $X \in L_r$ that*

$$\int \sup_{n\in\mathbb{N}} |P_s^{\mathscr{L}_n}X|^r\, dP \leqslant \delta \cdot u_{r,s} \int |X|^r\, dP,$$

where $u_{r,s} = (2^{|s-2|}\,[r/(r-s+1)])^{r/(s-1)}$ and $\delta = 1$ if all $\mathscr{L}_n$ are σ-fields or if X is non-negative and $\delta = 2$ elsewhere.

Proof. First we prove the assertion for $X \geqslant 0$. We use the following well known lemma [12, p. 231]:

If ξ, η: $\Omega \to [0, \infty]$ are $\mathscr{A}$-measurable with

$$P\{\eta > \varepsilon\} \leqslant (1/\varepsilon) \int_{\{\eta > \varepsilon\}} \xi \, dP \qquad \text{for all} \quad \varepsilon > 0, \tag{*}$$

then $P[\eta^\alpha] \leqslant [\alpha/(\alpha - 1)]^\alpha P[\xi^\alpha]$ for all $\alpha > 1$. Let $\eta := (\sup_{n \in \mathbb{N}} P_s^{\mathscr{L}_n} X)^{s-1}$, $\xi = 2^{|s-2|} X^{s-1}$, $\varepsilon = a^{s-1}$ and $\alpha = r/(s-1) > 1$. Then (*) is fulfilled according to Theorem 5.1. Hence the lemma cited above implies the assertion for $X \geqslant 0$. Hence the assertion is true for σ-fields and arbitrary $X \in L_r$, using $|P_s^{\mathscr{L}_n} X| \leqslant P_s^{\mathscr{L}_n} |X|$.

For σ-lattices and arbitrary $X \in L_r$ the assertion follows, because, according to (P12) of Section 3,

$$\sup_{n \in \mathbb{N}} |P_s^{\mathscr{L}_n} X|^r \leqslant \sup_{n \in \mathbb{N}} (P_s^{\mathscr{L}_n} |X|)^r + \sup_{n \in \mathbb{N}} (P_s^{\bar{\mathscr{L}}_n} |X|)^r.$$

We remark that for the case $s = 2$ and σ-fields we obtain in Theorem 5.1 and Corollary 5.2 exactly the classical maximal inequalities with the same bounding constants. For σ-fields $\mathscr{L}_n$ and $X \in L_s$ Ando and Amemiya [1] proved that

$$P\left[\sup_{n \in \mathbb{N}} |P_s^{\mathscr{L}_n} X|^s\right] \leqslant a_s P[|X|^s].$$

The constants a_s given by Ando and Amemiya are much larger than our constants $u_{r,s}$ specialized to the case $r = s$. For $s = 2$ we have $u_{s,s} = 4$ and $a_s = 208$ and for $s > 2$ there holds

$$(2^s u_{s,s})^{s-1} \leqslant a_s.$$

Observe that $1 \leqslant u_{s,s} = (2^{|s-2|} s)^{s/(s-1)} \to \infty$ with $s \to \infty$. For σ-lattices the preceding results of this section seem to be the first in this direction.

COROLLARY 5.3. *Let $(\Omega, \mathscr{A}, P)$ be a probability space, $1 < s < \infty$ and $r \geqslant s - 1$.* Let $\mathscr{L}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, *be a sequence of σ-lattices increasing or decreasing to the σ-lattice $\mathscr{L}_\infty$. Then for each $X \in L_r$ we have*

$$\|P_s^{\mathscr{L}_n} X - P_s^{\mathscr{L}_\infty} X\|_r \xrightarrow[n \in \mathbb{N}]{} 0.$$

Proof. According to Theorem 4.1 and Corollary 5.2 the assertion follows for $r > s - 1$ by the Theorem of Lebesgue. For $r = s - 1$ it suffices, according to Theorem 4.1, to prove that $|P_s^{\mathscr{L}_n} X|^{s-1}$, $n \in \mathbb{N}$, is uniformly integrable. Since $|P_s^{\mathscr{L}_n} X|^{s-1} \leqslant (P_s^{\mathscr{L}_n} |X|)^{s-1} + (P_s^{\bar{\mathscr{L}}_n} |X|)^{s-1}$ and the sum of two uniformly integrable sequences is uniformly integrable (use e.g. [4, Korollar 20.3, p. 90]), we may w.l.o.g. assume that $X \geqslant 0$. Let $X_n = P^{\mathscr{L}_n} X$, $n \in \mathbb{N}$. We obtain from Lemma 7.2(i) and (ii), applied pointwise for $\omega \in \{X_n > \alpha\}$ to $a = X_n(\omega)$, $x = X(\omega)$—since $\int_{\{X_n > \alpha\}} (X - X_n)^{\underline{s-1}} \, dP = 0$ according to (P8) of Section 3—that

$$\int_{\{X_n > \alpha\}} X_n^{s-1} \, dP \leqslant 2^{|s-2|} \int_{\{X_n > \alpha\}} X^{s-1} \, dP, \qquad \alpha > 0. \qquad (+)$$

Since $\sup_{n \in \mathbb{N}} P\{X_n > \alpha\} \to_{\alpha \to \infty} 0$ by Theorem 4.1, we consequently obtain from (+) that X_n^{s-1}, $n \in \mathbb{N}$, is uniformly integrable.

6. Characterizations of $P_s^{\mathscr{L}}$

In this section we characterize the operators $P_s^{\mathscr{L}}$ for σ-fields and σ-lattices. We start with a characterization result for σ-fields and arbitrary $s > 1$. For $s = 2$ this leads to a characterization for classical conditional expectation operators which is a common generalization of the results of Bahadur [2], Douglas [10], Moy [15] and Pfanzagl [17].

Theorem 6.1. *Let $1 < s < \infty$ and $s - 1 \leqslant r \leqslant \infty$. Let T: $L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ be an operator which is*

(i) *homogeneous,*

(ii) *translation invariant,*

(iii) *idempotent,*

(iv) *monotone,*

(v) *s-expectation invariant.*

Then there exists a sub-σ-field $\mathscr{B} \subset \mathscr{A}$ such that $TX = P_s^{\mathscr{B}} X$ for all $X \in L_r$.

Proof. Let $\mathscr{B} := \{A \in \mathscr{A}: T1_A = 1_A\}$. Since T is homogeneous and translation invariant, $A \in \mathscr{A}$ implies $T1_{\bar{A}} = 1 - T1_A = 1 - 1_A = 1_{\bar{A}}$. Hence $\mathscr{B}$ is a σ-field according to Lemma 7.3(ii). Since $T, P_s^{\mathscr{B}}$: $L_r \to L_r$ are monotone continuous operators according to Lemma 7.3(i) and Property (P5) of Theorem 3.2 it suffices to show that

$$TX = P_s^{\mathscr{B}} X \qquad \text{for bounded } X.$$

We remark that according to (i), (ii) and (iv) TX is bounded if X is bounded.

Applying (iv) of Lemma 7.3 to Z and $-Z$ we obtain

$$\int (X - TX)^{\underline{s-1}} Z \, dP = 0 \qquad \text{for} \quad Z \in L_\infty(\mathscr{B}).$$

Hence we can apply Theorem 3.4 with $Y = TX$ and the strictly increasing function $\varphi(t) = t$, $t \in \mathbb{R}$. This yields $TX = P_s^{\mathscr{L}} X$ for bounded X.

Easy examples show that none of the five conditions in Theorem 6.1 can be omitted without compensation. Another characterization result for conditional s-means given a σ-field can be found in [14].

In the case $s = 2$ s-expectation invariance of an operator T means $\int TX\,dP = \int X\,dP$. Hence for $s = 2$ monotony and s-expectation-invariance of T trivially imply that T is *weak* $\|\ \|_1$*-reducing*; i.e.,

$$\|TX - TY\|_1 \leqslant \|X - Y\|_1 \qquad \text{if} \quad X \leqslant Y.$$

Furthermore each linear operator with $\|TX\|_1 \leqslant \|X\|_1$ is trivially weak $\|\ \|_1$-reducing. Therefore the following corollary contains the characterization results for conditional expectation operators given in [2], [10], [15] and [17].

COROLLARY 6.2. *Let* $1 \leqslant r \leqslant \infty$ *and* $T: L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ *be an operator which is*

(i) *homogeneous,*

(ii) *translation invariant,*

(iii) *idempotent,*

(iv) *weak* $\|\ \|_1$*-reducing.*

Then there exists a sub-σ-field $\mathscr{B} \subset \mathscr{A}$ *such that* $TX = P_2^{\mathscr{B}} X$ *for all* $X \in L_r$.

Proof. According to Lemma 7.4 T is monotone and 2-expectation invariant. Hence Theorem 6.1, applied to $s = 2$, implies the assertion.

The only characterization result, known to the authors, concerning σ-lattices is the result of Dykstra [11] for the case $s = 2$. He shows that an operator $T: L_2(\Omega, \mathscr{A}, P) \to L_2(\Omega, \mathscr{A}, P)$ which is

(i) positive homogeneous,

(ii) $\|\ \|_2$-reducing; i.e., $\|TX - TY\|_2 \leqslant \|X - Y\|_2$,

(iii) idempotent,

(iv) monotone,

(v) 2-expectation invariant; i.e., $\int TX\,dP = \int X\,dP$,

(vi) 2-strictly monotonic at 0; i.e., $\int XTX\,dP > 0$ if $TX \neq 0$

is the conditional expectation operator $P_2^{\mathscr{L}}$ with respect to a suitable σ-lattice $\mathscr{L} \subset \mathscr{A}$.

We remark that Dykstra requires instead of (vi) a slightly stronger condition, but he uses in his proof only condition (vi) [3, p. 322]. Dykstra furthermore gives an example showing that conditions (i)–(v) above are not sufficient for his characterization result [3, p. 326].

In the characterization results for conditional expectation operators given a σ-field, only *one* integration condition besides algebraic conditions is used. This is an advantage since the authors believe that algebraic conditions are nicer and easier to verify than integration conditions. Dykstra's characterization result for a conditional expectation operator, given a σ-lattice, uses three integration conditions. The "strongest" of his integration conditions is the property (ii), namely, $\| \|_2$-reducing. In the following theorem we show that $\| \|_2$-reducing can be replaced by the algebraic condition translation-invariance. Furthermore monotony and 2-expectation invariance is weakened to weak $\| \|_1$-reducing. Since $\| \|_2$-reducing and $\| \|_2$-expectation invariance imply translation invariance (see [3], Proposition 7.2, 2°, p. 324), the theorem below contains the result of Dykstra.

THEOREM 6.3. *Let* $1 \leqslant r \leqslant \infty$ *and* T: $L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ *be an operator which is*

(i) *positive homogeneous,*

(ii) *translation invariant,*

(iii) *idempotent,*

(iv) *weak* $\| \|_1$*-reducing,*

(v) *2-strictly monotonic at* 0 *for* $X \in L_\infty$.

Then there exists a σ*-lattice* $\mathscr{L} \subset \mathscr{A}$ *such that* $TX = P_2^{\mathscr{L}} X$ *for all* $X \in L_r$.

Proof. Since according to (iv) T is $\| \|_1$-continuous for monotone sequences, it suffices to prove $TX = P_2^{\mathscr{L}} X$ for $X \in L_\infty$. According to Lemma 7.4 the operator is 2-expectation invariant and monotone and hence according to Lemma 7.3 there exists a σ-lattice $\mathscr{L} \subset \mathscr{A}$ such that

$$\int (X - TX) Z \, dP \leqslant 0 \qquad \text{for} \quad X \in L_\infty(\mathscr{A}),\ Z \in L_\infty(\mathscr{L}). \tag{1}$$

According to Theorem 3.4, applied to $s = 2$, $X := X - TX$, $Y = 0$ and $\varphi(t) = t$, (1) implies that

$$P_2^{\mathscr{L}}(X - T(X)) = 0. \tag{2}$$

Furthermore we have

$$T(X - P_2^{\mathscr{L}} X) = 0 \tag{3}$$

because $T(X - P_2^{\mathscr{L}} X) \in L_\infty(\mathscr{L})$ implies $\int (X - P_2^{\mathscr{L}} X)\, T(X - P_2^{\mathscr{L}} X)\, dP \leqslant 0$ and (v) implies $\int (X - P_2^{\mathscr{L}} X)\, T(X - P_2^{\mathscr{L}} X)\, dP > 0$ if $T(X - P_2^{\mathscr{L}} X) \neq 0$.

Now we shall show that

$$\|TX\|_1 = \|P_2^{\mathscr{L}} X\|_1, \qquad X \in L_\infty. \tag{4}$$

According to Lemma 7.4, the operator T is $\| \|_1$-reducing. Using (2), (3) and the fact that $P_2^{\mathscr{L}}$ is $\| \|_1$-reducing, too, we obtain:

$$\|P_2^{\mathscr{L}} X\|_1 = \|X - (X - P_2^{\mathscr{L}} X)\|_1 \geqslant \|TX - T(X - P_2^{\mathscr{L}} X)\|_1 = \|TX\|_1$$

and

$$\|TX\|_1 = \|X - (X - TX)\|_1 \geqslant \|P_2^{\mathscr{L}} X - P_2^{\mathscr{L}}(X - TX)\|_1 = \|P_2^{\mathscr{L}} X\|_1.$$

Hence (4) follows. Now (4) and translation invariance of T and $P_2^{\mathscr{L}}$ imply $\|P_2^{\mathscr{L}} X - a\|_1 = \|TX - a\|_1$ for $X \in L_\infty$, $a \in \mathbb{R}$; i.e., $\int |x-a| \, P_1(dx) = \int |x-a| \, P_2(dx)$ for $a \in \mathbb{R}$, where P_1 is the distribution of $P_2^{\mathscr{L}} X$ and P_2 the distribution of TX. Hence according to Lemma 7.5, $P_1 = P_2$ and therefore especially

$$\int (TX)^2 \, dP = \int (P_2^{\mathscr{L}} X)^2 \, dP. \tag{5}$$

Now we obtain by (5) and (1), that

$$\begin{aligned} \|TX - P_2^{\mathscr{L}} X\|_2^2 &= 2 \int (P_2^{\mathscr{L}} X)^2 \, dP - 2 \int TXP_2^{\mathscr{L}} X \, dp \\ &= 2 \left[\int XP_2^{\mathscr{L}} X \, dP - \int TX \, P_2^{\mathscr{L}} X \, dP \right] \\ &= 2 \int (X - TX) \, P_2^{\mathscr{L}} X \, dP \leqslant 0 \end{aligned}$$

and hence $TX = P_2^{\mathscr{L}} X$.

Dykstra's example cited in [3, p. 326] ($\Omega = \{1, 2\}$, $P\{1\} = P\{2\} = \frac{1}{2}$, $TX = X$ if $X(1) \leqslant X(2)$ and $TX(1) = X(2)$, $TX(2) = X(1)$ if $X(1) > X(2)$), shows that conditions (i)–(iv) of Theorem 6.3 are not sufficient to characterize conditional expectations operators given a σ-lattice. Now we give a characterization result for conditional s-means, given a σ-lattice. This is the first characterization of the operator $P_s^{\mathscr{L}}$ for $s \neq 2$ and σ-lattices $\mathscr{L}$.

THEOREM 6.4. *Let $1 < s < \infty$ and $s - 1 \leqslant r \leqslant \infty$. Let T: $L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ be an operator which is*

(i) *positive homogeneous,*

(ii) *translation invariant,*

(iii) *idempotent,*

(iv) *monotone,*

(v) *s-expectation invariant,*

(vi) *weak s-monotonic at* 0; *i.e.,* $\int X^{\underline{s-1}} TX\,dP \geq 0$ *for* $X \in L_\infty$ *with* $TX \geq 0$ *and fulfills*:

(vii) $T(X - \frac{1}{2}TX) = \frac{1}{2}TX.$

Then there exists a σ-lattice $\mathscr{L} \subset \mathscr{A}$ *such that* $TX = P_s^{\mathscr{L}} X$ *for all* $X \in L_r$.

Proof. According to Lemma 7.3(ii), (iii) we have that $\mathscr{L} := \{A \in \mathscr{A}: T1_A = 1_A\}$ is a σ-lattice and $TX \in L_r(\mathscr{L})$ as T is idempotent. Since T, $P_s^{\mathscr{L}}: L_r \to L_r$ are monotone continuous operators, it suffices to show that

$$TX = P_s^{\mathscr{L}} X \qquad \text{for bounded } X.$$

For this, it suffices to prove condition (i) and (ii) of Theorem 3.4 applied to the bounded function $Y = TX$. Condition (i) of Theorem 3.4 is fulfilled according to Lemma 7.3(iv). Since TX is bounded and $\mathscr{L}$-measurable, Condition (ii) of Theorem 3.4 is fulfilled with $\varphi(t) = t$ if we show that

$$\int (X - TX)^{\underline{s-1}} TX\,dP \geq 0. \tag{*}$$

Define inductively $\gamma_1 = \frac{1}{2}$ and $\gamma_{n+1} = (1 + \gamma_n)/2$. We prove by induction

$$T(X - \gamma_n TX) = (1 - \gamma_n) TX. \tag{**}$$

For $n = 1$ this is our Assumption (vii). Assume now that (**) holds for n. Then

$$\begin{aligned} T(X - \gamma_{n+1} TX) &= T(X - \gamma_n TX - \tfrac{1}{2}T(X - \gamma_n TX)) \\ &= \tfrac{1}{2}T(X - \gamma_n TX) = \tfrac{1}{2}(1 - \gamma_n) TX = (1 - \gamma_{n+1}) TX. \end{aligned}$$

Hence (**) holds for $(n+1)$. Let $X \geq 0$, then $T(X - \gamma_n TX) = (1 - \gamma_n) TX \geq 0$. Hence, according to (vi) and (**),

$$\begin{aligned} 0 &\leq \int (X - \gamma_n TX)^{\underline{s-1}} T(X - \gamma_n TX)\,dP \\ &= (1 - \gamma_n) \cdot \int (X - \gamma_n TX)^{\underline{s-1}} TX\,dP; \end{aligned}$$

i.e.,

$$\int (X - \gamma_n TX)^{\underline{s-1}} TX\,dP \geq 0.$$

As $\gamma_n \to 1$, we obtain (*) for bounded $X \geqslant 0$. As T is translation invariant we obtain (*) for all bounded X.

We remark that conditions (i)–(vi) are even in the case $s=2$ not sufficient to characterize s-means. This can be seen from the just cited example of Dykstra.

7. Auxiliary Lemmata

At first we prove a lemma characterizing a system of functions as a system of $\mathscr{L}$-measurable functions where $\mathscr{L}$ is a suitable σ-lattice. This lemma plays on analogous role for σ-lattices as Lemma 3 in [17] for σ-fields.

Lemma 7.1. *Let $0 < r \leqslant \infty$ and $\varnothing \neq F \subset L_r(\Omega, \mathscr{A}, P)$. Assume that F fulfills the following conditions:*

(i) $\alpha X + \beta \in F$ *for* $X \in F$, $\alpha \geqslant 0$, $\beta \in \mathbb{R}$,

(ii) $X \wedge Y$, $X \vee Y \in F$ *for* $X, Y \in F$,

(iii) $X_n \in F$, $X \in L_r$ *and* $X_n \uparrow X$ *or* $X_n \downarrow X$ *imply* $X \in F$.

Let $\mathscr{L} := \{A \in \mathscr{A}: 1_A \in F\}$. Then $\mathscr{L}$ is a σ-lattice and $F = L_r(\mathscr{L})$.

Proof. Since $1_{C_1 \cap C_2} = 1_{C_1} \wedge 1_{C_2}$, $1_{C_1 \cup C_2} = 1_{C_1} \vee 1_{C_2}$ and

$$1_{\bigcap_{k=k}^{n} C_k} \downarrow 1_{\bigcap_{k=1}^{\infty} C_k}, \qquad 1_{\bigcup_{k=1}^{n} C_k} \uparrow 1_{\bigcup_{k=1}^{\infty} C_k},$$

(ii), (iii) directly imply that $\mathscr{L}$ is a σ-lattice. Now we show $F \subset L_r(\mathscr{L})$. Let $X \in F$ be given then for each $\beta \in \mathbb{R}$

$$Y_n := [n(X-\beta) \vee 0] \wedge 1] \uparrow 1_{\{\omega: X(\omega) > \beta\}}.$$

As (i), (ii) imply $Y_n \in F$ we obtain from (iii) that $\{\omega: X(\omega) > \beta\} \in \mathscr{L}$; i.e., $X \in L_r(\mathscr{L})$.

Finally we have to show $L_r(\mathscr{L}) \subset F$. Let $X \in L_r(\mathscr{L})$. By (i) and (iii) we may w.l.o.g. assume that $X \geqslant 0$. Since $X \in L_r(\mathscr{L})$ we have

$$\{\omega: X(\omega) \geqslant \alpha\} \in \mathscr{L} \text{ and hence } 1_{\{\omega: X(\omega) \geqslant \alpha\}} \in F.$$

By (i) and (ii) this implies

$$X_n := \sup \left\{ \frac{\nu}{2^n} \cdot 1_{\{\omega: X(\omega) \geqslant \frac{\nu}{2^n}\}} : \nu = 0, 1, \ldots, n2^n \right\} \in F.$$

As $X_n \uparrow X$, hence $X \in F$ by (iii).

LEMMA 7.2. *Let* $1 < s < \infty$. *Then there hold the following inequalities between real numbers*:

(i) $a^{s-1} + 2^{s-2}(x-a)^{\underline{s-1}} \leqslant 2^{s-2}x^{\underline{s-1}}$ *if* $a \geqslant 0$, $s \geqslant 2$,

(ii) $a^{s-1} + (x-a)^{\underline{s-1}} \leqslant 2^{2-s}x^{s-1}$ *if* $a, x \geqslant 0$, $1 < s < 2$,

(iii) $|a - b \vee d|^s + |c - b \wedge d|^s \leqslant |a-b|^s + |c-d|^s$ *if* $a \geqslant c$.

Proof. Dividing (i) by a^{s-1} it suffices to prove

$$f(z) := 1 + 2^{s-2}(z-1)^{\underline{s-1}} \leqslant 2^{s-2}z^{\underline{s-1}} = g(z), \qquad z \in \mathbb{R}. \qquad (*)$$

Since $f(1) \leqslant g(1)$ and $f'(z) \leqslant g'(z)$ for $z \geqslant 1$, (*) is true for $z \geqslant 1$, $s > 2$. This directly implies that (*) is also true for $z \leqslant 0$. If $0 < z < 1$ use differential calculus to show that $h(z) = g(z) - f(z)$ attains its minimum at $z_0 = \frac{1}{2}$ and $h(\frac{1}{2}) = 0$.

Inequality (ii) follows in the same manner.

Inequality (iii) follows using similar techniques.

LEMMA 7.3. *Let* $1 < s < \infty$ *and* $s - 1 \leqslant r \leqslant \infty$. *Let* T: $L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ *be a positive homogeneous, translation invariant, monotone and s-expectation invariant operator. Then*

(i) *T is monotone continuous,*

(ii) $\mathscr{L} = \{A \in \mathscr{A}: T1_A = 1_A\}$ *is a σ-lattice,*

(iii) $L_r(\mathscr{L}) = \{X \in L_r: TX = X\}$,

(iv) $\int (X - TX)^{\underline{s-1}} Z \, dP \leqslant 0$ *for all* $Z \in L_\infty(\mathscr{L})$, $X \in L_r$.

Proof. (i) We show only that $X_n \uparrow X$ implies $TX_n \uparrow TX$ if $X_n, X \in L_r$. The decreasing case runs similarly. Since T is monotone the pointwise limit $Y := \lim_{n \to \infty} TX_n$ exists and fulfills $Y \leqslant TX$. As T is s-expectation invariant we obtain

$$\int (X_n - TX_n)^{\underline{s-1}} \, dP = 0$$

and hence by the Theorem of Lebesgue

$$\int (X - Y)^{\underline{s-1}} \, dP = 0.$$

Together with $Y \leqslant TX$ and $\int (X - TX)^{\underline{s-1}} \, dP = 0$ this implies

$$Y = TX.$$

For (ii), (iii), let $F := \{X \in L_r\colon\ TX = X\}$. Since T is positive homogeneous and translation invariant, condition (i) of Lemma 7.1 is fulfilled. To prove condition (ii) of Lemma 7.1 let X, $Y \in F$ be given. We show $X \vee Y \in F$, the proof for $X \wedge Y \in F$ runs similarly. Since T is monotone we have

$$T(X \vee Y) \geqslant TX \vee TY = X \vee Y.$$

As T is s-expectation invariant we have

$$0 = \int (X \vee Y - T(X \vee Y))^{\underline{s-1}}\, dP,$$

whence $T(X \vee Y) = X \vee Y$; i.e., $X \vee Y \in F$.

Condition (iii) of Lemma 7.1 follows from the monotone continuity of T. Hence Lemma 7.1 implies (ii), (iii).

Now we prove (iv) if $0 \leqslant X \leqslant 1$ and $Z = 1_C$ for $C \in \mathscr{L}$. Since T is monotone we have

$$T(X1_C) \leqslant TX1_C.$$

As T is s-expectation invariant we obtain

$$\begin{aligned} 0 &= \int (X1_C - T(X1_C))^{\underline{s-1}}\, dP \\ &\geqslant \int (X1_C - (TX)\,1_C)^{\underline{s-1}}\, dP \\ &= \int 1_C (X - TX)^{\underline{s-1}}\, dP. \end{aligned}$$

As T is positive homogeneous and translation invariant we obtain (iv) for all bounded X and $Z = 1_C$, $C \in \mathscr{L}$. As T is continuous on monotone sequences we obtain (iv) for all $X \in L_r(\mathscr{A})$ and $Z = 1_C$ with $C \in \mathscr{L}$. Since every non-negative $\mathscr{L}$-measurable function is monotone limit of $\mathscr{L}$-measurable simple functions and since T is s-expectation invariant we obtain (iv).

LEMMA 7.4. *Let* $1 \leqslant r \leqslant \infty$ *and* $T\colon L_r(\Omega, \mathscr{A}, P) \to L_r(\Omega, \mathscr{A}, P)$ *be a weak* $\|\ \|_1$*-reducing, translation invariant operator with* $T(0) = 0$. *Then* T *is*

(i) *monotone,*

(ii) *2-expectation invariant,*

(iii) $\|\ \|_1$*-reducing; i.e.,* $\|TX - TY\|_1 \leqslant \|X - Y\|_1$.

Proof. At first we show that T is 2-expectation invariant. Since $T0 = 0$ we obtain that

$$\|TX\|_1 \leqslant \|X\|_1 \qquad \text{if} \quad X \geqslant 0 \text{ or } X \leqslant 0. \tag{1}$$

Now let $X \in L_1(\mathscr{A})$ with $0 \leqslant X \leqslant c$. Since $c \leqslant |TX| + |c - TX|$ we obtain using (1) and translation invariance that

$$\begin{aligned} c &\leqslant \int [|TX| + |c - TX|]\, dP = \|TX\|_1 + \|T(X-c)\|_1 \\ &\leqslant \|X\|_1 + \|c - X\|_1 = c. \end{aligned} \tag{2}$$

Hence the two inequalities in (2) are equalities. The first of them implies $c = |TX| + |c - TX|$ and hence $0 \leqslant TX \leqslant c$, therefore the second implies $\int TX\, dP = \|TX\|_1 = \|X\|_1 = \int X\, dP$.

Since T is translation invariant, 2-expectation invariance holds for all bounded X. Since T is weakly $\| \,\|_1$-reducing we obtain that

$$\|TX_n - TX\|_1 \to 0 \qquad \text{if} \quad X_n \uparrow X \text{ or } X_n \downarrow X. \tag{3}$$

Let $X \in L_1(\mathscr{A})$ be bounded from below. Then there exist bounded functions X_n with $X_n \uparrow X$. Since $\int X_n\, dP = \int TX_n\, dP$, we obtain from (3)

$$\begin{aligned} \left| \int X\, dP - \int TX\, dP \right| &= \lim_{n \in \mathbb{N}} \left| \int X_n\, dP - \int TX\, dP \right| \\ &= \lim_{n \in \mathbb{N}} \left| \int TX_n\, dP - \int TX\, dP \right| \\ &\leqslant \lim_{n \in \mathbb{N}} \|TX_n - TX\|_1 = 0. \end{aligned}$$

Hence $\int X\, dP = \int TX\, dP$ for all $X \in L_r(\mathscr{A})$ which are bounded from below. Since each integrable function is a decreasing limit of integrable functions which are bounded from below, we obtain in a similar manner that $\int TX\, dP = \int X\, dP$ for all $X \in L_r(\mathscr{A})$, i.e., (ii).

Now we prove that T is monotone. Let $X, Y \in L_r(\mathscr{A})$ with $X \leqslant Y$. Since T is 2-expectation invariant and weak $\| \,\|_1$-reducing we obtain

$$\begin{aligned} \int (Y - X)\, dP = \int (TY - TX)\, dP \leqslant \int |TY - TX|\, dP = \|TY - TX\|_1 \\ \leqslant \|Y - X\|_1 = \int (Y - X)\, dP. \end{aligned}$$

Hence we have equality, whence $TY - TX = |TY - TX|$; i.e., $TX \leqslant TY$. Since T is monotone and weak $\| \|_1$-reducing, T is $\| \|_1$-reducing:

$$\|TX - TY\|_1 \leqslant \|T(X \vee Y) - T(X \wedge Y)\|_1 \leqslant \|X \vee Y - X \wedge Y\|_1 = \|X - Y\|_1.$$

LEMMA 7.5. *Let P_1, P_2 be two p-measures on the Borel-field of $\mathbb{R}$. If*

$$\int |x-a| \, P_1(dx) = \int |x-a| \, P_2(dx)$$

for all $a \in \mathbb{R}$, then $P_1 = P_2$.

Proof. It suffices to prove $P_1(-\infty, y] = P_2(-\infty, y]$ for all $y \in \mathbb{R}$. For all $\varepsilon > 0$ we have by assumption

$$\int (|x-y| - |x-(y+\varepsilon)|) \, P_1(dx) = \int (|x-y| - |x-(y+\varepsilon)|) \, P_2(dx);$$

i.e.,

$$\varepsilon[-P_1(-\infty, y] + P_1[y+\varepsilon, \infty)] + \int_{(y,\, y+\varepsilon)} (|x-y| - |x-(y+\varepsilon)|) \, P_1(dx)$$

$$= \varepsilon[-P_2(-\infty, y] + P_2[y+\varepsilon, \infty)] + \int_{(y,\, y+\varepsilon)} (|x-y| - |x-(y+\varepsilon)|) \, P_2(dx).$$

Divide by $\varepsilon > 0$ and let $\varepsilon \to 0$. Then we obtain

$$-P_1(-\infty, y] + P_1(y, \infty) = -P_2(-\infty, y] + P_2(y, \infty)$$

and hence

$$-2P_1(-\infty, y] - 1 = -2P_2(-\infty, y] - 1;$$

i.e., our assertion.

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